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Solution for Boundedness, Time Period and Graphs of Rate Measures in Matlab of Undamped and Unforced Duffing Equation

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ABSTRACT

The duffing equation is $\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t)$ is a non -linear second order differential equation. In this paper my aim is to solve for boundedness and time period of the duffing equation (undamped ($\delta = 0$) and unforced / undriven ($\gamma = 0$)) by Jacobi elliptic functions cn and nc, nd, cd and dc and sn. Also I expressed the identities, the properties and graphs using MATLAB program of three Jacobi elliptic functions. I observed the three special cases solve for boundedness and time period. They are Case A : solve Cubic $\nu \neq 0$ (special case ($\rho = \nu = 1$)), Case B : Solve Initial Value Problem for Cubic duffing equation with Special Cases for boundedness and time period which cannot be solved by cosine function. So, it can be solved in the form of $x(t) = x_0\varphi(\omega_0 t, m_0)$ in terms of Jacobi elliptic functions cn and nc, dc and cd, nd having positive frequency ω_0 and modulus m_0 in the interval [0, 1]. Case C : Solve Initial Value Problem for Linear ($\nu = 0$). Also for practical and research purposes I introduced the graphs of velocity and acceleration of duffing equation (undamped ($\delta = 0$) and unforced / undriven ($\gamma = 0$)) using MATLAB.

Keywords : Duffing Equation, Undamped And Unforced, Oscillators, Jacobi Elliptic Functions, Time Period, Boundedness

I. INTRODUCTION

The concept of Duffing equation was named after Georg Duffing (1861 – 1944) which is a non -linear second order differential equation in the case of damped ($\delta \neq 0$) and driven ($\gamma \neq 0$) equation.

The duffing equation is of the form

$$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t) \tag{1.1}$$

where $x(t)$ = displacement at time t

$\dot{x} = \frac{d}{dt}(x(t))$ = velocity

and $\ddot{x} = \frac{d^2}{dt^2}(x(t))$ = acceleration.

The numbers $\delta, \rho, \nu, \gamma, \omega$ are parameters.

Note that the motion of a damped ($\delta \neq 0$) and unforced / undriven ($\gamma = 0$) oscillator has more complex potential than Simple harmonic motion.

A. Parameters

The parameters of equation (1.1) are

- δ depends on the amount of damping
- ρ depends on the linear case
- ν depends on the non-linear case
- γ is the amplitude of the periodic driving force
- ω is the angular frequency of the periodic driving force

B. Equilibrium Points

The force provided for the non-linear case is

$\nu x^3 + \rho x$ (1.2)

Case – I

If $\rho > 0$ and $\nu > 0$ then (1.2) is called a hardening spring and the equilibrium point is at $x = 0$

Case – II

If $\rho > 0$ and $\nu < 0$ then (1.2) is called a softening spring and the equilibrium points are at $x = \pm\sqrt{\frac{\rho}{\nu}}$

Case - III

If $\rho < 0$ and $\nu > 0$ then (1.2) is also called a softening spring and the equilibrium points are at $x = \pm\sqrt{-\frac{\rho}{\nu}}$

C. Methods of Solution

Many approximate solutions for the duffing equation are

- By Fourier series method.
- By Frobenius method which yields a complex solution.
- By Euler's method and Runge-Kutta methods in numerical analysis.
- By Homotopy analysis method which yields approximate solutions of the duffing equation.
- By Jacobi Elliptic functions to obtain the exact solutions of the undamped ($\delta = 0$) and undriven ($\gamma = 0$) duffing equation.

II. UNDAMPED ($\delta = 0$) AND UNFORCED ($\gamma = 0$) DUFFING EQUATION BY JACOBI ELLIPTIC FUNCTIONS

The duffing equation is

$$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma cos\,(\omega t) \qquad (2.1)$$

In this paper, my aim is to solve for boundedness and its time period of the duffing equation (undamped ($\delta = 0$) and unforced / undriven ($\gamma = 0$)) by Jacobi elliptic functions cn and nc, dn and nd, cd and dc, sn. (see [2, 5-8, 11])

Also I expressed the identities, the properties and graphs using MATLAB program of Jacobi elliptic functions. (see [1])

By taking the plus sign in (2.1) becomes

$$\ddot{x} + \nu x^3 + \rho x = 0 \qquad (2.2)$$

In this paper there are three special cases to solve for boundedness and time period. These are

(i) For cubic case ($\nu \neq 0$) (special case ($\nu = 1 = \rho$)) without initial conditions

(ii) For cubic case ($\nu \neq 0$) with initial conditions

(iii) For linear case ($\nu = 0$) with initial conditions

A. Properties of Jacobi Elliptic Functions

(i) Jacobi elliptic functions are $sn(t,\, m) = sin\theta, cn(t,\, m) cos\theta, dn(t,\, m) = \sqrt{1 - m^2\, sin^2\,\theta}$ where $m \in (0,\, 1)$ is called elliptic modulus, $t = \int_0^\theta \frac{d\phi}{\sqrt{1-m^2\,sin^2\,\phi}}$ is the time period and $\theta = am(t,\, m)$ is called Jacobi amplitude.

(ii) Jacobi elliptic functions are derivable

(iii) Graphs using MATLAB (see [1])

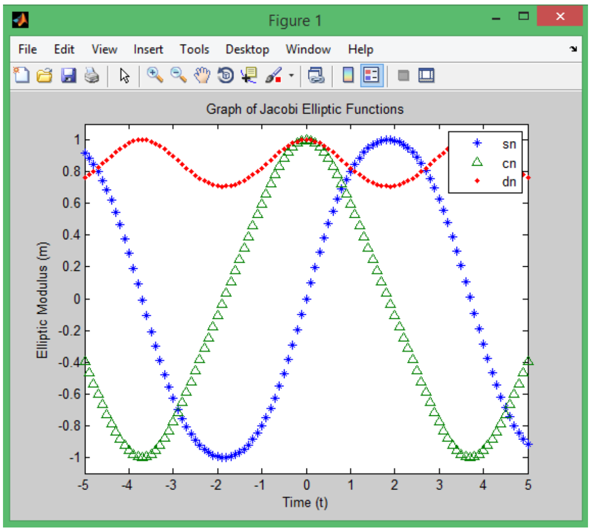
Program :

```
>> m = 0.5;
>> t = -5 : 0.1 : 5;
>> [s, c, d] = ellipj(t, m);

>> plot(t, s, '*', t, c, '^', t,
 d, '.')
>> axis([-5 5 -1.1 1.1])
>> legend('sn', 'cn', 'dn')
>> xlabel('Time (t)')
>> ylabel('Elliptic Modulus (m)')
```

>> title('Graph of Jacobi Elliptic Functions')

Graph :



(iv) sn and cn are periodic functions period of sn = $4K\left(\frac{1}{4}\right)$ = 4K(m), m = $\frac{1}{4}$ period of cn = $4K\left(\frac{1}{4}\right)$ = 4K(m), m = $\frac{1}{4}$

where K = K(m) = $K\left(\frac{1}{4}\right) \approx 1.5962422$

B. Identities of Jacobi Elliptic Functions

- $sn^2(t,m) + cn^2(t,m) = 1$
- $dn^2(t,m) = 1 - m^2\, sn^2(t,m)$
- $\lim_{m\to 0} sn(t,m) = sint = sn(t,0)$
- $\lim_{m\to 0} cn(t,m) = cost = cn(t,0)$
- $\lim_{m\to 0} dn(t,m) = 1 = dn(t,0)$
- $\lim_{m\to 1} sn(t,m) = tanht = sn(t,1)$
- $\lim_{m\to 1} cn(t,m) = cosht = cn(t,1)$
- $\lim_{m\to 1} dn(t,m) = secht = dn(t,1)$
- $\frac{d}{dt} sn(t,m) = cn(t,m)dn(t,m)$
- $\frac{d}{dt} cn(t,m) = -sn(t,m)dn(t,m)$
- $\frac{d}{dt} dn(t,m) = -m^2 sn(t,m)cn(t,m)$

III. SOLVE CUBIC FOR BOUNDEDNESS AND TIME PERIOD

Case A : For Cubic $\nu \neq 0$ (Special case ($\rho = 1 = \nu$))

From (1.1) the undamped ($\delta = 0$) and unforced ($\gamma = 0$) duffing equation is

$$\ddot{x} + \rho x + \nu x^3 = 0$$
$$\Rightarrow \dot{x}(\ddot{x} + \rho x + \nu x^3) = 0$$
$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}(\dot{x})^2 + \frac{1}{2}\rho x^2 + \frac{1}{4}\nu x^4\right] = 0$$
$$\Rightarrow \frac{1}{2}(\dot{x})^2 + \frac{1}{2}\rho x^2 + \frac{1}{4}\nu x^4 = H \qquad (3.1)$$

is called invariant of motion and where H is called the Hamiltonian Operator. where H is to be determined by putting the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ Here since ρ and ν are positive then the solution of (3.1) is bounded.

If $|x| \leq \sqrt{\frac{2H}{\rho}}$ and $|\dot{x}| \leq \sqrt{2H}$ then the Hamiltonian H is positive.

Now Substituting $\rho = 1 = \nu$ in (2.2) and (3.1), we have $\ddot{x} + x^3 + x = 0$ and $\frac{1}{2}(\ddot{x})^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H$ which transforms to Hamiltonian system of first order differential equations. (see [4], [10])

$$\dot{x} = y \qquad (3.2)$$
$$\dot{y} = -x - x^3 \qquad (3.3)$$

and $\frac{1}{2}(\dot{x})^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H$ (3.4)

Now we have to find out the time period t by solving (3.4)

$$(\dot{x})^2 = 2H - x^2 - \frac{1}{2}x^4$$
$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2H - x^2 - \frac{1}{2}x^4$$
$$\Rightarrow \frac{dx}{dt} = \sqrt{2H - x^2 - \frac{1}{2}x^4}$$

$$\Rightarrow \int dt = \int \frac{dx}{\sqrt{2H - x^2 - \frac{1}{2}x^4}}$$

$\Rightarrow t = \int \frac{dx}{\sqrt{2H-x^2-\frac{1}{2}x^4}}$ is the Time Period.

From (3.4)

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H(x,y)$$

$$\Rightarrow \frac{\partial H}{\partial x} = x + x^2 = -\dot{y} = -\ddot{x}$$

$$\Rightarrow \frac{\partial H}{\partial y} = y = \dot{x}$$

The Hamiltonian system of equations (see [3]) are

$$\dot{x} = \frac{\partial H}{\partial y}$$

$$\dot{y} = -\frac{\partial H}{\partial x}$$

Case B : Solve Initial Value Problem (IVP) for Cubic with different Special Cases

Solve the initial value problem $\ddot{x}(t) + \rho x(t) + \nu x^3(t) = 0, x(0) = x_0 \; and \; \dot{x}(0) = \dot{x}_0$ which is a second order non-linear differential equation by using Jacobi elliptic function.

Solution :

Setting $x(t) = c_1 cn(\omega t + c_2, m)$ (3.5)

satisfies the equation (2.2).

Now ω and m are to be determined in terms of ρ and ν and c_1, c_2 are to be determined from initial conditions $x(0) = x_0 \; and \; \dot{x}(0) = \dot{x}_0$

Now $\ddot{x}(t) + \omega^2(1 - 2m^2)x(t) + \frac{2m^2\omega^2}{c_1^2} x^3(t)\; 0,$

$$x(0) = x_0 \; and \; \dot{x}(0) = \dot{x}_0 \tag{3.6}$$

where c_1, c_2 are constants.

Now, comparing (2.2) and (3.6), we get

$$\rho = \omega^2(1 - 2m^2) \; and \; \nu = \frac{2m^2\omega^2}{c_1^2}$$

$$\Rightarrow \rho = \omega^2 - 2m^2\omega^2 \; and \; \nu c_1^2 = 2m^2\omega^2$$

$$\Rightarrow \omega^2 = \rho + \nu c_1^2$$

$$\Rightarrow \omega = \sqrt{\rho + \nu c_1^2} \tag{3.7}$$

and $\nu c_1^2 = 2m^2\omega^2$

$$\Rightarrow \nu c_1^2 = 2m^2(\rho + \nu c_1^2) \; (\because \; \omega^2 = \rho + \nu c_1^2)$$

$$\Rightarrow m^2 = \frac{\nu c_1^2}{2(\rho + \nu c_1^2)}$$

$$\Rightarrow m = \sqrt{\frac{\nu c_1^2}{2(\rho + \nu c_1^2)}} \tag{3.8}$$

Now, substituting the values of (3.7) and (3.8) in (3.5),

$$\text{we get } x(t) = c_1 cn\left(\left(\sqrt{\rho + c_1^2\nu}\right)t + c_2, \sqrt{\frac{c_1^2\nu}{2(\rho + c_1^2\nu)}}\right) \tag{3.9}$$

$$\Rightarrow x(0) = c_1 cn\left(\left(\sqrt{\rho + c_1^2\nu}\right) \times 0 + c_2, \sqrt{\frac{c_1^2\nu}{2(\rho + c_1^2\nu)}}\right)$$

$$= c_1 cn(c_2, m)$$

$$= x_0$$

$$\text{and } \dot{x}(0) = c_1 \left[\begin{matrix} -sn\left(\left(\sqrt{\rho + c_1^2\nu}\right) \times 0 + c_2, m\right) \\ dn\left(\left(\sqrt{\rho + c_1^2\nu}\right) \times 0 + c_2, m\right) \end{matrix}\right] \sqrt{\rho + c_1^2\nu}$$

$$= -\sqrt{\rho + c_1^2\nu}\; c_1 sn(c_2, m) dn(c_2, m)$$

$$= \dot{x}_0$$

Now I have to discuss about the solution by different cases to the given problem in the form of $x(t) = x_0\varphi(\omega_0 t, m_0)$ in terms of Jacobi elliptic functions nc, dc and cd, nd having positive frequency ω_0 and modulus m_0 on the interval [0, 1].

Case i : $\dot{x}(0) = 0$

Then – $\sqrt{\rho + c_1^2\nu}\; c_1 sn(c_2, m) dn(c_2, m) = 0$

$$\Rightarrow c_1 = 0 = x_0 \tag{3.10}$$

$$\therefore c_2 = 0 \tag{3.11}$$

Now the solution to the initial value problem

$$\ddot{x}(t) + \rho x(t) + \nu x^3(t) = 0, x(0) = x_0 \; and \; \dot{x}(0) = 0 \tag{3.12}$$

is

$$x(t) = x_0 cn\left(\sqrt{\rho + \nu x_0^2}\, t, \sqrt{\frac{\nu x_0^2}{2(\rho+\nu x_0^2)}}\right), \rho + \nu x_0^2 \neq 0 \quad (3.13)$$

Here the required solution is bounded.

Substituting $\rho = 1 = \nu$ in (3.12), we get the time period.

Case ii : $x(0) = 1, \dot{x}(0) = 0, \rho = \nu = 2$

Now, equation (2.2) becomes

$$\ddot{x} + 2x + 2x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

Now, $x(t) = 1 \times cn\left(\sqrt{2 + 2 \times 1^2}\, t, \sqrt{\frac{2\times 1^2}{2(2+2\times 1^2)}}\right)$

$\Rightarrow x(t) = cn\left(2t, \frac{1}{2}\right)$ positive root can be taken.

The solution is bounded since $\nu = \rho = 2$ and $\omega_0 = 2$ which is positive.

Its period is $\frac{4K\left(\frac{1}{2}\right)}{2}$ from Jacobi function cn.

$$\text{Let } \vartheta = \rho + \nu x_0^2 = \omega^2 \quad (3.14)$$

$$\text{and } \mu = \frac{\nu x_0^2}{2(\rho+\nu x_0^2)} = m^2 \quad (3.15)$$

Then (3.5) becomes $x(t) = x_0 cn\left(\sqrt{\vartheta}\, t, \sqrt{\mu}\right)$ (3.16)

Case a : $\vartheta < 0$ or $\mu < 0$

Now, (3.16) becomes (see [2])

$x(t) = x_0 nc\left(\sqrt{-\vartheta}\, t, \sqrt{1-\mu}\right), \vartheta < 0$

and $0 < \mu \leq 1$ (3.17)

Case b : $0 < \mu < 1, \rho = \nu = -2, x(0) = 1, \dot{x}(0) = 0$ from (3.17)

Now, equation (2.2) becomes

$$\ddot{x} - 2x - 2x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

Now, $\vartheta = (-2) + (-2)1^2$ and $\mu = \frac{(-2)1^2}{2(-2+(-2)1^2)}$

$= -4$ and $= \frac{1}{4}$

$$\therefore x(t) = nc\left(\sqrt{-4}\, t, \sqrt{1 - \frac{1}{4}}\right)$$

$= nc\left(2t, \frac{\sqrt{3}}{2}\right)$

Again, $x(t) = 1 \times$

$cn\left(\sqrt{(-2) + (-2)1^2}\, t, \sqrt{\frac{(-2)1^2}{2((-2)+(-2)1^2)}}\right) =$

$cn\left(2\sqrt{-1}t, \frac{1}{2}\right)$

$$\therefore x(t) = nc\left(2t, \frac{\sqrt{3}}{2}\right) = cn\left(2\sqrt{-1}t, \frac{1}{2}\right)$$

The solution is unbounded since $\nu = \rho = -2$ and $\omega_0 = 2\sqrt{-1}$ which is negative.

Its period is $\frac{4K\left(\frac{\sqrt{3}}{2}\right)}{2}$ from Jacobi function nc.

Case c : $\mu = 1, \rho = 1, \nu = -2, x(0) = 1, \dot{x}(0) = 0$ from (3.17)

Now, equation (2.2) becomes

$$\ddot{x} + x - 2x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\nu = -2, \rho = 1$ and $\omega_0 = \sqrt{-1}$ which is negative.

Its period is $4K(0)$ from Jacobi function nc.

Case d : $\vartheta < 0$ and $\mu > 1$

Now, (3.16) becomes (see [2])

$x(t) = x_0 dc\left(\sqrt{-\vartheta\mu}t, \sqrt{1 - \frac{1}{\mu}}\right), \vartheta < 0$ and $\mu > 1$

(3.18)

Case e : $\vartheta < 0, \mu > 1, \rho = 3, \nu = -1, x(0) = 2, \dot{x}(0) = 0$ from (3.18)

Now, equation (2.2) becomes

$$\ddot{x} + 3x - x^3 = 0, x(0) = 2 \text{ and } \dot{x}(0) = 0$$

which is an initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\rho = 3, \nu = -1$ and $\omega_0 = \sqrt{2}$ which is positive.

Its period is $2 \times \frac{4K\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{2}}$ from Jacobi function dc.

Case f : $\vartheta > 0$ and $\mu < 0$

Now, (3.16) becomes (see [9])

$$x(t) = x_0 cd\left(\sqrt{\vartheta(1-\mu)}t, \frac{\sqrt{-\mu}}{\sqrt{1-\mu}}\right), \vartheta > 0 \quad \text{and} \quad \mu < 0 \qquad (3.19)$$

Case g : $\vartheta > 0, \mu < 0$, $\rho = 2, \nu = -1, x(0) = 1, \dot{x}(0) = 0$ from (3.19)

Now, equation (2.2) becomes

$$\ddot{x} + 2x - x^3 = 0, x(0) = 1 \; and \; \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\rho = 2, \nu = -1$ and $\omega_0 = \sqrt{\frac{3}{2}}$ which is positive.

Its period is $\frac{4K\left(\frac{\sqrt{3}}{3}\right)}{\sqrt{\frac{3}{2}}}$ from Jacobi function cd.

Case h : $\vartheta < 0$ and $\mu < 0$

Now, (3.16) becomes (see [9])

$$x(t) = x_0 nd\left(\sqrt{-\vartheta(1-\mu)}t, \frac{1}{\sqrt{1-\mu}}\right), \vartheta < 0 \quad \text{and} \quad \mu < 0 \qquad (3.20)$$

Case i : $\vartheta < 0$, $\mu < 0$, $\rho = -2, \nu = 13, x(0) = 10^{-2}, \dot{x}(0) = 0$ from (3.20)

Now, equation (2.2) becomes

$$\ddot{x} - 2x + 13x^3 = 0, x(0) = 10^{-2} \; and \; \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\rho = -2, \nu = 13$ and $\omega_0 = 1.4139$ which is positive.

Its period is $0.01 \times \frac{4K(0.99985)}{1.4139}$ from Jacobi function nd.

Case C : Solve Initial Value Problem for Linear $\nu = 0$

Equation (2.2) becomes $\ddot{x}(t) + \rho x(t) = 0$ which is a second order linear differential equation.

Now, solve the initial value problem $\ddot{x}(t) + \rho x(t) = 0, x(0) = x_0, \dot{x}(0) = \dot{x}_0$.

Solution : The auxiliary equation of $\ddot{x}(t) + \rho x(t) = 0$ is

$$M^2 + \rho = 0$$

$$\Rightarrow M = 0 \pm i\sqrt{\rho}$$

$$\therefore \; x(t) = e^{0\times t}\left(c_1 cos\sqrt{\rho}\, t + c_2 sin\sqrt{\rho}\, t\right)$$

$$= \left(c_1 cos\sqrt{\rho}\, t + c_2 sin\sqrt{\rho}\, t\right),$$

c_1, c_2 are constants.

$$\Rightarrow x(0) = \left(c_1 cos\sqrt{\rho} \times 0 + c_2 sin\sqrt{\rho} \times 0\right)$$

$$\Rightarrow x_0 = c_1$$

Now $\dot{x}(t) = \left(-c_1 sin\sqrt{\rho}\, t + c_2 cos\sqrt{\rho}\, t\right)$

$$\Rightarrow \dot{x}(0) = 0$$

$$\Rightarrow \dot{x}_0 = c_2 = 0$$

$$\therefore \; x(t) = x_0 cos\sqrt{\rho}\, t \qquad (3.21)$$

is the required particular solution which is bounded and its period is to be determined by suitable value of ρ.

IV. GRAPHS OF VELOCITY AND ACCELERATION OF UNDAMPED AND UNDRIVEN DUFFING EQUATION USING

MATLAB The concept of Duffing equation was named after Georg Duffing (1861 – 1944) which is a non -linear second order differential equation in the case of damped $(\delta \neq 0)$ and driven $(\gamma \neq 0)$ equation.

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The numbers $\delta, \rho, \nu, \gamma, \omega$ are parameters.

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The parameters of equation (1.1) are

- δ depends on the amount of damping
- ρ depends on the linear case
- ν depends on the non-linear case
- γ is the amplitude of the periodic driving force
- ω is the angular frequency of the periodic driving force

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The force provided for the non-linear case is

$$\nu x^3 + \rho x \qquad (1.2)$$

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If $\rho > 0$ and $\nu > 0$ then (1.2) is called a hardening spring and the equilibrium point is at $x = 0$

Case – II

If $\rho > 0$ and $\nu < 0$ then (1.2) is called a softening spring and the equilibrium points are at $x = \pm\sqrt{\frac{\rho}{\nu}}$

Case - III

If $\rho < 0$ and $\nu > 0$ then (1.2) is also called a softening spring and the equilibrium points are at $x = \pm\sqrt{-\frac{\rho}{\nu}}$

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Many approximate solutions for the duffing equation are

- By Fourier series method.
- By Frobenius method which yields a complex solution.
- By Euler's method and Runge-Kutta methods in numerical analysis.
- By Homotopy analysis method which yields approximate solutions of the duffing equation.
- By Jacobi Elliptic functions to obtain the exact solutions of the undamped ($\delta = 0$) and undriven ($\gamma = 0$) duffing equation.

V. UNDAMPED ($\delta = 0$) AND UNFORCED ($\gamma = 0$) DUFFING EQUATION BY JACOBI ELLIPTIC FUNCTIONS

The duffing equation is

$$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t) \tag{2.1}$$

In this paper, my aim is to solve for boundedness and its time period of the duffing equation (undamped ($\delta = 0$) and unforced / undriven ($\gamma = 0$)) by Jacobi elliptic functions cn and nc, dn and nd, cd and dc, sn. (see [2, 5-8, 11])

Also I expressed the identities, the properties and graphs using MATLAB program of Jacobi elliptic functions. (see [1])

By taking the plus sign in (2.1) becomes

$$\ddot{x} + \nu x^3 + \rho x = 0 \tag{2.2}$$

In this paper there are three special cases to solve for boundedness and time period. These are

(i) For cubic case ($\nu \neq 0$) (special case ($\nu = 1 = \rho$)) without initial conditions

(ii) For cubic case ($\nu \neq 0$) with initial conditions

(ii) For linear case ($\nu = 0$) with initial conditions

A. Properties of Jacobi Elliptic Functions

(i) Jacobi elliptic functions are $sn(t,\, m) = sin\theta, cn(t,\, m) cos\theta,\ dn(t,\, m) = \sqrt{1 - m^2\sin^2\theta}$

where $m \in (0,\,1)$ is called elliptic modulus,

$t = \int_0^\theta \frac{d\phi}{\sqrt{1-m^2\sin^2\phi}}$ is the time period

and $\theta = am(t,\, m)$ is called Jacobi amplitude.

(ii) Jacobi elliptic functions are derivable

(iii) Graphs using MATLAB (see [1])

Program :

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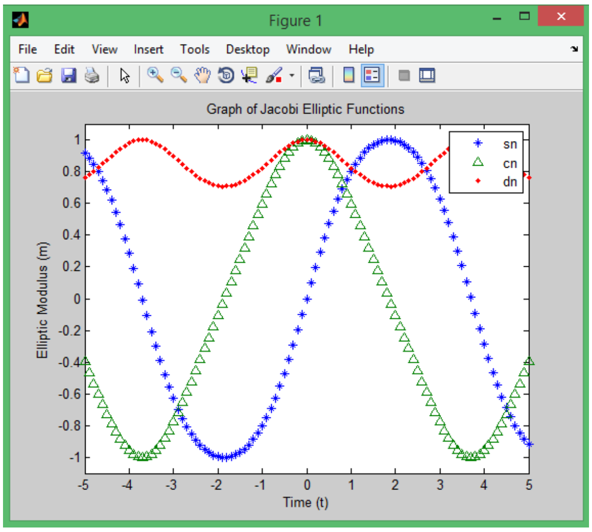
 d, '.')
>> axis([-5 5 -1.1 1.1])
>> legend('sn', 'cn', 'dn')
>> xlabel('Time (t)')
>> ylabel('Elliptic Modulus (m)')
>> title('Graph of Jacobi
 Elliptic Functions')
```

Graph :



(iv) sn and cn are periodic functions

period of sn = $4K\left(\frac{1}{4}\right) = 4K(m),\ m = \frac{1}{4}$

period of cn = $4K\left(\frac{1}{4}\right) = 4K(m),\ m = \frac{1}{4}$

where $K = K(m) = K\left(\frac{1}{4}\right) \approx 1.5962422$

B. Identities of Jacobi Elliptic Functions

- $sn^2(t,m) + cn^2(t,m) = 1$
- $dn^2(t,m) = 1 - m^2\,\mathrm{sn}^2(t,m)$
- $\lim_{m\to 0} sn(t,m) = sint = sn(t,0)$

- $\lim_{m\to 0} cn(t,m) = cost = cn(t,0)$
- $\lim_{m\to 0} dn(t,m) = 1 = dn(t,0)$
- $\lim_{m\to 1} sn(t,m) = tanht = sn(t,1)$
- $\lim_{m\to 1} cn(t,m) = cosht = cn(t,1)$
- $\lim_{m\to 1} dn(t,m) = secht = dn(t,1)$
- $\frac{d}{dt} sn(t,m) = cn(t,m)dn(t,m)$
- $\frac{d}{dt} cn(t,m) = -sn(t,m)dn(t,m)$
- $\frac{d}{dt} dn(t,m) = -m^2 sn(t,m)cn(t,m)$

VI. SOLVE CUBIC FOR BOUNDEDNESS AND TIME PERIOD

Case A : For Cubic $\nu \neq 0$ (Special case ($\rho = 1 = \nu$))

From (1.1) the undamped ($\delta = 0$) and unforced ($\gamma = 0$) duffing equation is

$$\ddot{x} + \rho x + \nu x^3 = 0$$

$$\Rightarrow \dot{x}(\ddot{x} + \rho x + \nu x^3) = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}(\dot{x})^2 + \frac{1}{2}\rho x^2 + \frac{1}{4}\nu x^4\right] = 0$$

$$\Rightarrow \frac{1}{2}(\dot{x})^2 + \frac{1}{2}\rho x^2 + \frac{1}{4}\nu x^4 = H \qquad (3.1)$$

is called invariant of motion and where H is called the Hamiltonian Operator. where H is to be determined by putting the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x_0}$ Here since ρ and ν are positive then the solution of (3.1) is *bounded*.

If $|x| \leq \sqrt{\frac{2H}{\rho}}$ and $|\dot{x}| \leq \sqrt{2H}$ then the Hamiltonian H is positive.

Now Substituting $\rho = 1 = \nu$ in (2.2) and (3.1),

we have $\ddot{x} + x^3 + x = 0$ and $\frac{1}{2}(\ddot{x})^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H$

which transforms to Hamiltonian system of first order differential equations. (see [4], [10])

$$\dot{x} = y \qquad (3.2)$$

$$\dot{y} = -x - x^3 \qquad (3.3)$$

and $\frac{1}{2}(\dot{x})^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H \qquad (3.4)$

Now we have to find out the time period t by solving (3.4)

$$(\dot{x})^2 = 2H - x^2 - \frac{1}{2}x^4$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 2H - x^2 - \frac{1}{2}x^4$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{2H - x^2 - \frac{1}{2}x^4}$$

$$\Rightarrow \int dt = \int \frac{dx}{\sqrt{2H - x^2 - \frac{1}{2}x^4}}$$

$\Rightarrow t = \int \frac{dx}{\sqrt{2H - x^2 - \frac{1}{2}x^4}}$ is the Time Period.

From (3.4)

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H(x,y)$$

$$\Rightarrow \frac{\partial H}{\partial x} = x + x^2 = -\dot{y} = -\ddot{x}$$

$$\Rightarrow \frac{\partial H}{\partial y} = y = \dot{x}$$

The Hamiltonian system of equations (see [3]) are

$$\dot{x} = \frac{\partial H}{\partial y}$$

$$\dot{y} = -\frac{\partial H}{\partial x}$$

Case B : Solve Initial Value Problem (IVP) for Cubic with different Special Cases

Solve the initial value problem $\ddot{x}(t) + \rho x(t) + \nu x^3(t) = 0$, $x(0) = x_0$ *and* $\dot{x}(0) = \dot{x}_0$ which is a second order non-linear differential equation by using Jacobi elliptic function.

Solution :

Setting $x(t) = c_1 cn(\omega t + c_2, m) \qquad (3.5)$

satisfies the equation (2.2).

Now ω and m are to be determined in terms of ρ and ν and c_1, c_2 are to be determined from initial conditions $x(0) = x_0$ *and* $\dot{x}(0) = \dot{x}_0$

Now $\ddot{x}(t) + \omega^2(1-2m^2)x(t) + \frac{2m^2\omega^2}{c_1^2}x^3(t)\ 0,$

$$x(0) = x_0 \ and \ \dot{x}(0) = \dot{x}_0 \qquad (3.6)$$

where c_1, c_2 are constants.

Now, comparing (2.2) and (3.6), we get

$$\rho = \omega^2(1-2m^2) \ and \ \nu = \frac{2m^2\omega^2}{c_1^2}$$

$$\Rightarrow \rho = \omega^2 - 2m^2\omega^2 \ and \ \nu c_1^2 = 2m^2\omega^2$$

$$\Rightarrow \omega^2 = \rho + \nu c_1^2$$

$$\Rightarrow \omega = \sqrt{\rho + \nu c_1^2} \qquad (3.7)$$

and $\nu c_1^2 = 2m^2\omega^2$

$$\Rightarrow \nu c_1^2 = 2m^2(\rho + \nu c_1^2) \quad (\because \ \omega^2 = \rho + \nu c_1^2)$$

$$\Rightarrow m^2 = \frac{\nu c_1^2}{2(\rho + \nu c_1^2)}$$

$$\Rightarrow m = \sqrt{\frac{\nu c_1^2}{2(\rho + \nu c_1^2)}} \qquad (3.8)$$

Now, substituting the values of (3.7) and (3.8) in (3.5), we get

$$x(t) = c_1 cn\left(\left(\sqrt{\rho + c_1^2\nu}\right)t + c_2, \sqrt{\frac{c_1^2\nu}{2(\rho + c_1^2\nu)}}\right) \qquad (3.9)$$

$$\Rightarrow x(0) = c_1 cn\left(\left(\sqrt{\rho + c_1^2\nu}\right) \times 0 + c_2, \quad \sqrt{\frac{c_1^2\nu}{2(\rho + c_1^2\nu)}}\right)$$

$$= c_1 cn(c_2, m)$$

$$= x_0$$

and $\dot{x}(0) = c_1 \begin{bmatrix} -sn\left(\left(\sqrt{\rho + c_1^2 \nu}\right) \times 0 + c_2, m\right) \\ dn\left(\left(\sqrt{\rho + c_1^2 \nu}\right) \times 0 + c_2, m\right) \end{bmatrix} \sqrt{\rho + c_1^2 \nu}$

$= -\sqrt{\rho + c_1^2 \nu}\, c_1 sn(c_2, m) dn(c_2, m)$

$= \dot{x}_0$

Now I have to discuss about the solution by different cases to the given problem in the form of $x(t) = x_0 \varphi(\omega_0 t,\ m_0)$ in terms of Jacobi elliptic functions nc, dc and cd, nd having positive frequency ω_0 and modulus m_0 on the interval [0, 1].

Case i : $\dot{x}(0) = 0$

Then $-\sqrt{\rho + c_1^2 \nu}\, c_1 sn(c_2, m) dn(c_2, m) = 0$

$\Rightarrow\ c_1 = 0 = x_0$ (3.10)

$\therefore\ c_2 = 0$ (3.11)

Now the solution to the initial value problem

$$\ddot{x}(t) + \rho x(t) + \nu x^3(t) = 0,\ x(0) = x_0\ and\ \dot{x}(0) = 0 \quad (3.12)$$

is

$$x(t) = x_0 cn\left(\sqrt{\rho + \nu x_0^2}\, t,\ \sqrt{\frac{\nu x_0^2}{2(\rho + \nu x_0^2)}}\right),\ \rho + \nu x_0^2 \neq 0 \quad (3.13)$$

Here the required solution is bounded.

Substituting $\rho = 1 = \nu$ in (3.12), we get the time period.

Case ii : $x(0) = 1, \dot{x}(0) = 0,\ \rho = \nu = 2$

Now, equation (2.2) becomes

$$\ddot{x} + 2x + 2x^3 = 0, \qquad x(0) = 1\ \ and\ \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

Now, $x(t) = 1 \times cn\left(\sqrt{2 + 2 \times 1^2}\, t, \sqrt{\frac{2\times 1^2}{2(2+2\times 1^2)}}\right)$

$\Rightarrow x(t) = cn\left(2t, \frac{1}{2}\right)$ positive root can be taken.

The solution is bounded since $\nu = \rho = 2$ and $\omega_0 = 2$ which is positive.

Its period is $\frac{4K\left(\frac{1}{2}\right)}{2}$ from Jacobi function cn.

Let $\vartheta = \rho + \nu x_0^2 = \omega^2$ (3.14)

and $\mu = \frac{\nu x_0^2}{2(\rho + \nu x_0^2)} = m^2$ (3.15)

Then (3.5) becomes $x(t) = x_0 cn\left(\sqrt{\vartheta}\, t,\ \sqrt{\mu}\right)$ (3.16)

Case a : $\vartheta < 0$ or $\mu < 0$

Now, (3.16) becomes (see [2])

$x(t) = x_0 nc\left(\sqrt{-\vartheta}\, t,\ \sqrt{1-\mu}\right),\ \vartheta < 0$ and $0 < \mu \leq 1$ (3.17)

Case b : $0 < \mu < 1,\ \rho = \nu = -2,\ x(0) = 1,\ \dot{x}(0) = 0$ from (3.17)

Now, equation (2.2) becomes

$$\ddot{x} - 2x - 2x^3 = 0, \qquad x(0) = 1\ \ and\ \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

Now, $\vartheta = (-2) + (-2)1^2$ and $\mu = \frac{(-2)1^2}{2(-2+(-2)1^2)}$

$= -4$ and $= \frac{1}{4}$

$\therefore\ x(t) = nc\left(\sqrt{-4}\, t,\ \sqrt{1 - \frac{1}{4}}\right.$

$= nc\left(2t, \frac{\sqrt{3}}{2}\right)$

Again, $x(t) = 1 \times cn\left(\sqrt{(-2) + (-2)1^2}\, t, \sqrt{\frac{(-2)1^2}{2((-2)+(-2)1^2)}}\right) = cn\left(2\sqrt{-1}t,\ \frac{1}{2}\right)$

$\therefore\ x(t) = nc\left(2t, \frac{\sqrt{3}}{2}\right) = cn\left(2\sqrt{-1}t,\ \frac{1}{2}\right)$

The solution is unbounded since $\nu = \rho = -2$ and $\omega_0 = 2\sqrt{-1}$ which is negative.

Its period is $\frac{4K\left(\frac{\sqrt{3}}{2}\right)}{2}$ from Jacobi function nc.

Case c : $\mu = 1,\ \rho = 1,\ \nu = -2,\ x(0) = 1,\ \dot{x}(0) = 0$ from (3.17)

Now, equation (2.2) becomes

$$\ddot{x} + x - 2x^3 = 0, \qquad x(0) = 1\ \ and\ \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\nu = -2,\ \rho = 1$ and $\omega_0 = \sqrt{-1}$ which is negative.

Its period is $4K(0)$ from Jacobi function nc.

Case d : $\vartheta < 0\ and\ \mu > 1$

Now, (3.16) becomes (see [2])

$$x(t) = x_0 dc\left(\sqrt{-\vartheta\mu}t,\ \sqrt{1 - \frac{1}{\mu}}\right),\ \vartheta < 0 \text{ and } \mu > 1 \quad (3.18)$$

Case e : $\vartheta < 0, \mu > 1,\ \rho = 3,\ \nu = -1,\ x(0) = 2,\ \dot{x}(0) = 0$ from (3.18)

Now, equation (2.2) becomes

$$\ddot{x} + 3x - x^3 = 0, \qquad x(0) = 2\ \ and\ \dot{x}(0) = 0$$

which is an initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\rho = 3,\ \nu = -1$ and $\omega_0 = \sqrt{2}$ which is positive.

Its period is $2 \times \frac{4K\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{2}}$ from Jacobi function dc.

Case f : $\vartheta > 0$ and $\mu < 0$

Now, (3.16) becomes (see [9])

$$x(t) = x_0 cd\left(\sqrt{\vartheta(1-\mu)}t, \frac{\sqrt{-\mu}}{\sqrt{1-\mu}}\right), \ \vartheta > 0 \quad \text{and} \quad \mu < 0 \quad (3.19)$$

Case g : $\vartheta > 0$, $\mu < 0$, $\rho = 2$, $\nu = -1$, $x(0) = 1$, $\dot{x}(0) = 0$ from (3.19)

Now, equation (2.2) becomes

$$\ddot{x} + 2x - x^3 = 0, \quad x(0) = 1 \ \ and \ \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\rho = 2$, $\nu = -1$ and $\omega_0 = \sqrt{\frac{3}{2}}$ which is positive.

Its period is $\frac{4K\left(\frac{\sqrt{3}}{3}\right)}{\sqrt{\frac{3}{2}}}$ from Jacobi function cd.

Case h : $\vartheta < 0$ and $\mu < 0$

Now, (3.16) becomes (see [9])

$$x(t) = x_0 nd\left(\sqrt{-\vartheta(1-\mu)}t, \frac{1}{\sqrt{1-\mu}}\right), \ \vartheta < 0 \quad \text{and} \quad \mu < 0 \quad (3.20)$$

Case i : $\vartheta < 0$, $\mu < 0$, $\rho = -2$, $\nu = 13$, $x(0) = 10^{-2}$, $\dot{x}(0) = 0$ from (3.20)

Now, equation (2.2) becomes

$$\ddot{x} - 2x + 13x^3 = 0, \quad x(0) = 10^{-2} \ \ and \ \dot{x}(0) = 0$$

which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since $\rho = -2$, $\nu = 13$ and $\omega_0 = 1.4139$ which is positive.

Its period is $0.01 \times \frac{4K(0.99985)}{1.4139}$ from Jacobi function nd.

Case C : Solve Initial Value Problem for Linear $\nu = 0$

Equation (2.2) becomes $\ddot{x}(t) + \rho x(t) = 0$ which is a second order linear differential equation.

Now, solve the initial value problem $\ddot{x}(t) + \rho x(t) = 0$, $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$.

Solution : The auxiliary equation of $\ddot{x}(t) + \rho x(t) = 0$ is

$M^2 + \rho = 0$

$\Rightarrow M = 0 \pm i\sqrt{\rho}$

$\therefore \ x(t) = e^{0\times t}\left(c_1 cos\sqrt{\rho}\, t + c_2 sin\sqrt{\rho}\, t\right)$

$= \left(c_1 cos\sqrt{\rho}\, t + c_2 sin\sqrt{\rho}\, t\right)$,

$c_1, \ c_2$ are constants.

$\Rightarrow x(0) = \left(c_1 cos\sqrt{\rho} \times 0 + c_2 sin\sqrt{\rho} \times 0\right)$

$\Rightarrow x_0 = c_1$

Now $\dot{x}(t) = \left(-c_1 sin\sqrt{\rho}\, t + c_2 cos\sqrt{\rho}\, t\right)$

$\Rightarrow \dot{x}(0) = 0$

$\Rightarrow \dot{x}_0 = c_2 = 0$

$$\therefore \ x(t) = x_0 cos\sqrt{\rho}\, t \quad (3.21)$$

is the required particular solution which is bounded and its period is to be determined by suitable value of ρ.

IV. GRAPHS OF VELOCITY AND ACCELERATION OF UNDAMPED AND UNDRIVEN DUFFING EQUATION USING MATLAB

Consider the duffing equation

$$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\cos(\omega t) \quad (4.1)$$

$where \ x = x(t), t = time \ period$

Here $x(t)$ = displacement, $\dot{x}$ = velocity, $\ddot{x}$ = acceleration

The graphs of velocity and acceleration for duffing equation with suitable parameters $\rho = -2$, $\nu = 13$, $\gamma = 0$, $\delta = 0$, $\omega = 1.4139$ using MATLAB (see [1]) program are shown below where X-axis indicates Time (t) in second and Y-axis indicates Velocity and Acceleration Graph with respect to Time (t).

Program :

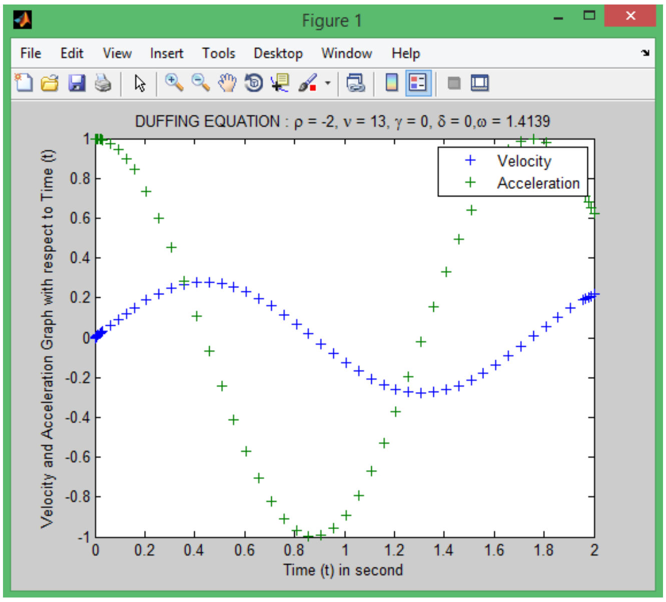
M-file

```
function xdiff = duffing_2(t,x)
rho = -2;
nu = 13;
gamma = 0;
delta = 0;
omega = 1.4139;
dxdt = [x(2); -delta*x(2)-rho*(x(1))-
    nu*(x(1)^3)+gamma*cos(omega*t)];
xdiff = dxdt;
return
```

Command Window

```
>> [t, x] = ode45(@duffing_2, [0 2],
    [0 1]);
>> plot(t, x, '+')
>> title('DUFFING EQUATION : \rho = -2,
 \nu = 13, \gamma = 0, \delta = 0,
 \omega = 1.4139')
>> xlabel('Time (t) in second')
>> ylabel('Velocity and Acceleration
 Graph with respect to Time (t)')
>> legend('Velocity', 'Acceleration')
```

Graph :



Consider the duffing equation

$$\ddot{x} + \delta\dot{x} + (\nu x^3 \pm \rho x) = \gamma\, cos(\omega t) \qquad (4.1)$$

where $x = x(t), t = time\ period$

Here $x(t) =$ displacement, $\dot{x} =$ velocity, $\ddot{x} =$ acceleration

The graphs of velocity and acceleration for duffing equation with suitable parameters $\rho = -2, \nu = 13, \gamma = 0, \delta = 0, \omega = 1.4139$ using MATLAB (see [1]) program are shown below where X-axis indicates Time (t) in second and Y-axis indicates Velocity and Acceleration Graph with respect to Time (t).

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omega = 1.4139;
dxdt = [x(2); -delta*x(2)-rho*(x(1))-
 nu*(x(1)^3)+gamma*cos(omega*t)];
xdiff = dxdt;
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```

Command Window

```
>> [t, x] = ode45(@duffing_2, [0 2],
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 \nu = 13, \gamma = 0, \delta = 0,
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>> ylabel('Velocity and Acceleration
 Graph with respect to Time (t)')
>> legend('Velocity', 'Acceleration')
```
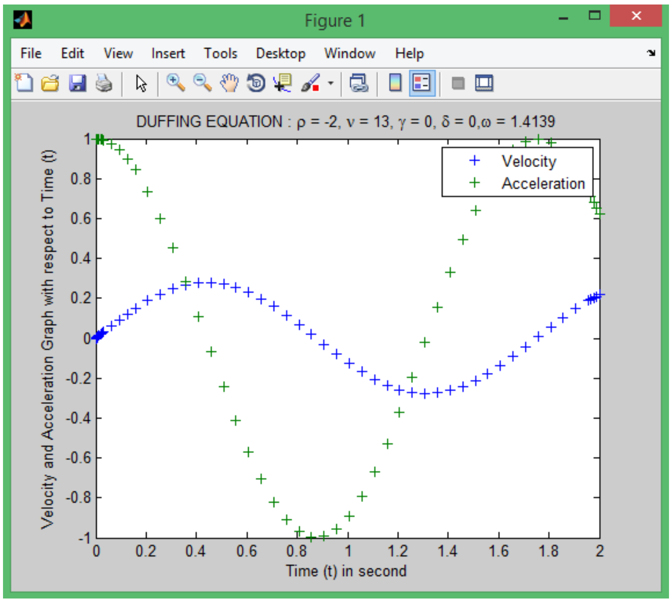

Graph :



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