Solution for Boundedness, Time Period and Graphs of Rate Measures in Matlab of Undamped and Unforced Duffing Equation

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ABSTRACT

The duffing equation is a non-linear second order differential equation. In this paper my aim is to solve for boundedness and time period of the duffing equation (undamped and unforced) by Jacobi elliptic functions cn and nc, nd and cd and dc and sn. Also I expressed the identities, the properties and graphs using MATLAB program of three Jacobi elliptic functions. I observed the three special cases solve for boundedness and time period. They are Case A : solve Cubic \( \nu \neq 0 \) (special case \( \rho = \nu = 1 \)), Case B : Solve Initial Value Problem for Cubic duffing equation with Special Cases for boundedness and time period which cannot be solved by cosine function. So, it can be solved in the form of \( x(t) = x_0 \phi(\omega_0 t, m_0) \) in terms of Jacobi elliptic functions cn and nc, dc and cd, nd having positive frequency \( \omega_0 \) and modulus \( m_0 \) in the interval \([0, 1]\). Case C : Solve Initial Value Problem for Linear \( \nu = 0 \). Also for practical and research purposes I introduced the graphs of velocity and acceleration of duffing equation (undamped and unforced) using MATLAB.

Keywords : Duffing Equation, Undamped And Unforced, Oscillators, Jacobi Elliptic Functions, Time Period, Boundedness

I. INTRODUCTION

The concept of Duffing equation was named after Georg Duffing (1861 – 1944) which is a non-linear second order differential equation in the case of damped \( \delta \neq 0 \) and driven \( \gamma \neq 0 \) equation.

The duffing equation is of the form

\[ \ddot{x} + \delta \dot{x} + (\nu x^3 \pm \rho x) = \gamma \cos(\omega t) \quad (1.1) \]

where \( x(t) \) = displacement at time \( t \)

\( \dot{x} = \frac{d}{dt}(x(t)) = \text{velocity} \)

and \( \ddot{x} = \frac{d^2}{dt^2}(x(t)) = \text{acceleration} \).

The numbers \( \delta, \rho, \nu, \gamma, \omega \) are parameters.

Note that the motion of a damped \( \delta \neq 0 \) and unforced / undriven \( \gamma = 0 \) oscillator has more complex potential than Simple harmonic motion.

A. Parameters

The parameters of equation (1.1) are

- \( \delta \) depends on the amount of damping
- \( \rho \) depends on the linear case
- \( \nu \) depends on the non-linear case
- \( \gamma \) is the amplitude of the periodic driving force
- \( \omega \) is the angular frequency of the periodic driving force
B. Equilibrium Points

The force provided for the non-linear case is
\[ \nu x^3 + \rho x \]  
(1.2)

**Case – I**
If \( \rho > 0 \) and \( \nu > 0 \) then (1.2) is called a hardening spring and the equilibrium point is at \( x = 0 \)

**Case – II**
If \( \rho > 0 \) and \( \nu < 0 \) then (1.2) is called a softening spring and the equilibrium points are at \( x = \pm \sqrt{-\frac{\rho}{\nu}} \)

**Case - III**
If \( \rho < 0 \) and \( \nu > 0 \) then (1.2) is also called a softening spring and the equilibrium points are at \( x = \pm \sqrt{-\frac{\rho}{\nu}} \)

C. Methods of Solution

Many approximate solutions for the duffing equation are
- By Fourier series method.
- By Frobenius method which yields a complex solution.
- By Euler’s method and Runge-Kutta methods in numerical analysis.
- By Homotopy analysis method which yields approximate solutions of the duffing equation.
- By Jacobi Elliptic functions to obtain the exact solutions of the undamped (\( \delta = 0 \)) and undriven (\( \gamma = 0 \)) duffing equation.

II. UNDAMPED (\( \delta = 0 \)) AND UNFORCED (\( \gamma = 0 \)) DUFFING EQUATION BY JACOBI ELLIPTIC FUNCTIONS

The duffing equation is
\[ \ddot{x} + \delta \dot{x} + (\nu x^3 \pm \rho x) = \gamma \cos(\omega t) \]  
(2.1)

In this paper, my aim is to solve for boundedness and its time period of the duffing equation (undamped (\( \delta = 0 \)) and unforced / undriven (\( \gamma = 0 \)) by Jacobi elliptic functions cn and nc, dn and nd, cd and dc, sn. (see [2, 5-8, 11])

Also I expressed the identities, the properties and graphs using MATLAB program of Jacobi elliptic functions. (see [1])

By taking the plus sign in (2.1) becomes
\[ \ddot{x} + \nu x^3 + \rho x = 0 \]  
(2.2)

In this paper there are three special cases to solve for boundedness and time period. These are
- For cubic case (\( \nu \neq 0 \)) (special case (\( \nu = 1 = \rho \)) without initial conditions
- For cubic case (\( \nu \neq 0 \)) with initial conditions
- For linear case (\( \nu = 0 \)) with initial conditions

A. Properties of Jacobi Elliptic Functions

(i) Jacobi elliptic functions are \( sn(t, m) = \sin\theta, cn(t, m)\cos\theta, dn(t, m) = \sqrt{1 - m^2 \sin^2\theta} \)
where \( m \in (0, 1) \) is called elliptic modulus, \( t = \int_{0}^{\theta} \frac{d\phi}{\sqrt{1 - m^2 \sin^2\phi}} \) is the time period and \( \theta = am(t, m) \) is called Jacobi amplitude.

(ii) Jacobi elliptic functions are derivable

(iii) Graphs using MATLAB (see [1])

Program :
\[
\begin{align*}
& \gg m = 0.5; \\
& \gg t = -5 : 0.1 : 5; \\
& \gg [s, c, d] = \text{ellipj}(t, m); \\
& \gg \text{plot}(t, s, '^', t, c, '^', t, d, '.'); \\
& \gg \text{axis([-5 5 -1.1 1.1])} \\
& \gg \text{legend('sn', 'cn', 'dn')} \\
& \gg \text{xlabel('Time (t)')} \\
& \gg \text{ylabel('Elliptic Modulus (m)')} 
\end{align*}
\]
Graph:

(iv) sn and cn are periodic functions period of \( sn = 4K(\frac{1}{4}) = 4K(m), m = \frac{1}{4} \) period of \( cn = 4K(\frac{1}{4}) = 4K(m), m = \frac{1}{4} \)

where \( K = K(m) = K(\frac{1}{4}) \approx 1.5962422 \)

B. Identities of Jacobi Elliptic Functions

- \( sn^2(t, m) + cn^2(t, m) = 1 \)
- \( dn^2(t, m) = 1 - m^2 sn^2(t, m) \)
- \( \lim_{m \to 0} sn(t, m) = \sin t = sn(t, 0) \)
- \( \lim_{m \to 0} cn(t, m) = \cos t = cn(t, 0) \)
- \( \lim_{m \to 0} dn(t, m) = 1 = dn(t, 0) \)
- \( \lim_{m \to 1} sn(t, m) = \tanh t = sn(t, 1) \)
- \( \lim_{m \to 1} cn(t, m) = \coth t = cn(t, 1) \)
- \( \lim_{m \to 1} dn(t, m) = 1 = dn(t, 1) \)
- \( \frac{d}{dt} sn(t, m) = cn(t, m)dn(t, m) \)
- \( \frac{d}{dt} cn(t, m) = -sn(t, m)dn(t, m) \)
- \( \frac{d}{dt} dn(t, m) = -m^2 sn(t, m)cn(t, m) \)

III. SOLVE CUBIC FOR BOUNDEDNESS AND TIME PERIOD

Case A : For Cubic \( \nu \neq 0 \) (Special case \( \rho = 1 = \nu \))

From (1.1) the undamped \( (\delta = 0) \) and unforced \( (\gamma = 0) \) duffing equation is

\[
\ddot{x} + \rho \dot{x} + \nu x^3 = 0
\]

\[
\Rightarrow \ddot{x}(\dot{x} + \rho \dot{x} + \nu x^3) = 0
\]

\[
\Rightarrow \frac{d}{dt} \left[ \frac{1}{2}(\dot{x})^2 + \frac{1}{2} \rho \dot{x}^2 + \frac{1}{4} \nu x^4 \right] = 0
\]

\[
\Rightarrow \frac{1}{2}(\dot{x})^2 + \frac{1}{2} \rho \dot{x}^2 + \frac{1}{4} \nu x^4 = H
\]

(3.1)

is called invariant of motion and where H is called the Hamiltonian Operator. where H is to be determined by putting the initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \) Here since \( \rho \) and \( \nu \) are positive then the solution of (3.1) is bounded.

If \( |x| \leq \frac{2H}{\sqrt{\rho}} \) and \( |\dot{x}| \leq \sqrt{2H} \) then the Hamiltonian H is positive.

Now Substituting \( \rho = 1 = \nu \) in (2.2) and (3.1), we have \( \ddot{x} + x^3 + x = 0 \) and \( \frac{1}{2}(\dot{x})^2 + \frac{1}{2} \dot{x}^2 + \frac{1}{4} x^4 = H \)

which transforms to Hamiltonian system of first order differential equations. (see [4], [10])

\[
\dot{x} = y
\]

(3.2)

\[
\dot{y} = -x - x^3
\]

(3.3)

and \( \frac{1}{2}(\dot{x})^2 + \frac{1}{2} \dot{x}^2 + \frac{1}{4} x^4 = H \)

(3.4)

Now we have to find out the time period \( t \) by solving (3.4)

\[
(\dot{x})^2 = 2H - x^2 - \frac{1}{2} x^4
\]

\[
\Rightarrow \left( \frac{dx}{dt} \right)^2 = 2H - x^2 - \frac{1}{2} x^4
\]

\[
\Rightarrow \frac{dx}{dt} = \sqrt{2H - x^2 - \frac{1}{2} x^4}
\]
\[ \int dt = \int \frac{dx}{\sqrt{2H - x^2 - \frac{1}{2}x^4}} \]
\[ \Rightarrow t = \int \frac{dx}{\sqrt{2H - x^2 - \frac{1}{2}x^4}} \text{ is the Time Period.} \]

From (3.4)
\[ \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 = H(x, y) \]
\[ \Rightarrow \frac{\partial H}{\partial x} = x + x^2 = -\dot{y} = -\ddot{x} \]
\[ \Rightarrow \frac{\partial H}{\partial y} = y = \dot{x} \]

The Hamiltonian system of equations (see [3]) are
\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial y} \\
\dot{y} &= -\frac{\partial H}{\partial x}
\end{align*}
\]

**Case B : Solve Initial Value Problem (IVP) for Cubic with different Special Cases**

Solve the initial value problem \( \ddot{x}(t) + \rho x(t) + \nu x^3(t) = 0, x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \) which is a second order non-linear differential equation by using Jacobi elliptic function.

**Solution :**

Setting \( x(t) = c_1 cn(\omega t + c_2, m) \) (3.5) satisfies the equation (2.2).

Now \( \omega \) and \( m \) are to be determined in terms of \( \rho \) and \( \nu \) and \( c_1, c_2 \) are to be determined from initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \)

Now \( \ddot{x}(t) + \omega^2 (1 - 2m^2) x(t) + \frac{2m^2 \omega^2}{c_1^2} x^3(t) = 0, \)
\[ x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0 \]
\[ (3.6) \]

where \( c_1, c_2 \) are constants.

Now, comparing (2.2) and (3.6), we get
\[ \rho = \omega^2 (1 - 2m^2) \text{ and } \nu = \frac{2m^2 \omega^2}{c_1^2} \]
\[ \Rightarrow \omega^2 = \rho - 2m^2 \omega^2 \text{ and } \nu c_1^2 = 2m^2 \omega^2 \]
\[ \Rightarrow \omega^2 = \rho + \nu c_1^2 \]
\[ \Rightarrow \omega = \sqrt{\rho + \nu c_1^2} \]
\[ (3.7) \]
and \( \nu c_1^2 = 2m^2 \omega^2 \)

\[ \Rightarrow \nu c_1^2 = 2m^2 (\rho + \nu c_1^2) (\because \omega^2 = \rho + \nu c_1^2) \]
\[ \Rightarrow m^2 = \frac{\nu c_1^2}{2(\rho + \nu c_1^2)} \]
\[ \Rightarrow m = \sqrt{\frac{\nu c_1^2}{2(\rho + \nu c_1^2)}} \]
\[ (3.8) \]

Now, substituting the values of (3.7) and (3.8) in (3.5),
we get
\[ x(t) = c_1 cn \left( \sqrt{\rho + \nu c_1^2} t + c_2, \sqrt{\frac{c_1^2 \nu}{2(\rho + \nu c_1^2)}} \right) \]
\[ (3.9) \]

\[ \Rightarrow x(0) = c_1 cn \left( \sqrt{\rho + \nu c_1^2} \times 0 + c_2, \frac{c_1^2 \nu}{2(\rho + \nu c_1^2)} \right) \]
\[ = c_1 cn(c_2, m) \]
\[ = x_0 \]

and \( \dot{x}(0) = c_1 \left[ -sn \left( \left( \sqrt{\rho + \nu c_1^2} \times 0 + c_2, m \right) \right) \right] \sqrt{\rho + \nu c_1^2} \]
\[ = -\sqrt{\rho + \nu c_1^2} \nu c_1 sn(c_2, m)dn(c_2, m) \]
\[ = \dot{x}_0 \]

Now I have to discuss about the solution by different cases to the given problem in the form of \( x(t) = x_0 \phi(\omega_0 t, m_0) \) in terms of Jacobi elliptic functions nc, dc and cd, nd having positive frequency \( \omega_0 \) and modulus \( m_0 \) on the interval [0, 1].

**Case i : \( \dot{x}(0) = 0 \)**

Then \( -\sqrt{\rho + \nu c_1^2} \nu c_1 sn(c_2, m)dn(c_2, m) = 0 \)
\[ \Rightarrow c_1 = 0 = x_0 \]
\[ (3.10) \]
\[ \Rightarrow c_2 = 0 \]
\[ (3.11) \]

Now the solution to the initial value problem
\[ \ddot{x}(t) + \rho x(t) + \nu x^3(t) = 0, x(0) = x_0 \text{ and } \dot{x}(0) = 0 \]
\[ (3.12) \]
is
\[ x(t) = x_0 cn \left( \sqrt{\rho + vx_0^2} t, \sqrt{\frac{vx_0^2}{(\rho + vx_0^2)}} \right), \quad \rho + vx_0^2 \neq 0 \]  
(3.13)

Here the required solution is bounded.

Substituting \( \rho = 1 = \nu \) in (3.12), we get the time period.

**Case ii:** \( x(0) = 1, \dot{x}(0) = 0, \rho = \nu = 2 \)

Now, equation (2.2) becomes
\[ \ddot{x} + 2x + 2x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0 \]

which is a initial value problem, whether it is bounded or not and find its period.

**Solution:**

Now, \( x(t) = 1 \times cn \left( \sqrt{2 + 2 \times 1^2} t, \sqrt{\frac{2 \times 1^2}{2(2+2 \times 1^2)}} \right) \)  
\[ \Rightarrow x(t) = cn \left( 2t \frac{1}{2} \right) \]  
positive root can be taken.

The solution is bounded since \( \nu = \rho = 2 \) and \( \omega_0 = 2 \) which is positive.

Its period is \( \frac{4K(\frac{1}{2})}{2} \) from Jacobi function \( cn \).

Let \( \theta = \rho + vx_0^2 = \omega^2 \)  
(3.14)

and \( \mu = \frac{vx_0^2}{2(\rho + vx_0^2)} = m^2 \)  
(3.15)

Then (3.5) becomes \( x(t) = x_0 cn \left( \sqrt{\theta} t, \sqrt{\mu} \right) \)  
(3.16)

**Case a:** \( \theta < 0 \) or \( \mu < 0 \)

Now, (3.16) becomes (see [2])
\[ x(t) = x_0 dc \left( -\sqrt{-\theta} t, \sqrt{1 - \mu} \right), \theta < 0 \]  
and \( 0 < \mu \leq 1 \)  
(3.17)

**Case b:** \( 0 < \mu < 1, \rho = \nu = -2, x(0) = 1, \dot{x}(0) = 0 \)

from (3.17)

Now, equation (2.2) becomes
\[ \ddot{x} - 2x - 2x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0 \]

which is a initial value problem, whether it is bounded or not and find its period.

**Solution:**

Now, \( \theta = (-2) + (-2)1^2 \) and \( \mu = \frac{(-2)1^2}{2(-2+(-2)1^2)} \)  
\[ = -4 \text{ and } = \frac{1}{4} \]

\[ x(t) = ncn \left( \sqrt{-4} t, \sqrt{1 - \frac{1}{4}} \right) = ncn \left( 2t, \frac{\sqrt{3}}{2} \right) \]

Again, \( x(t) = 1 \times \)
\[ cn \left( \sqrt{(-2) + (-2)1^2} t, \sqrt{\frac{(-2)1^2}{2((-2)+(-2)1^2)}} \right) = \]
\[ cn \left( 2\sqrt{1t}, \frac{1}{2} \right) \]

The solution is unbounded since \( \nu = \rho = -2 \) and \( \omega_0 = 2\sqrt{1} \) which is negative.

Its period is \( \frac{4K(\frac{1}{2})}{2} \) from Jacobi function \( nc \).

**Case c:** \( \mu = 1, \rho = 1, \nu = -2, x(0) = 1, \dot{x}(0) = 0 \)

from (3.17)

Now, equation (2.2) becomes
\[ \ddot{x} + x - 2x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0 \]

which is a initial value problem, whether it is bounded or not and find its period.

**Solution:**

The solution is unbounded since \( \nu = -2, \rho = 1 \) and \( \omega_0 = \sqrt{-1} \) which is negative.

Its period is \( 4K(0) \) from Jacobi function \( nc \).

**Case d:** \( \theta < 0 \) and \( \mu > 1 \)

Now, (3.16) becomes (see [2])
\[ x(t) = x_0 dc \left( -\sqrt{-\theta} t, \sqrt{1 - \mu} \right), \theta < 0 \text{ and } \mu > 1 \]  
(3.18)

**Case e:** \( \theta < 0, \mu > 1, \rho = 3, \nu = -1, x(0) = 2, \dot{x}(0) = 0 \)

from (3.18)

Now, equation (2.2) becomes
\[ \ddot{x} + 3x - x^3 = 0, x(0) = 2 \text{ and } \dot{x}(0) = 0 \]

which is an initial value problem, whether it is bounded or not and find its period.

**Solution:**

The solution is unbounded since \( \rho = 3, \nu = -1 \) and \( \omega_0 = \sqrt{2} \) which is positive.

Its period is \( 2 \times \frac{4K(\frac{\sqrt{2}}{2})}{\sqrt{2}} \) from Jacobi function \( dc \).

**Case f:** \( \theta > 0 \) and \( \mu < 0 \)
Now, (3.16) becomes (see [9])

\[ x(t) = x_0 cd \left( \sqrt{\theta(1 - \mu)} t, \frac{1}{\sqrt{1 - \mu}} \right), \theta > 0 \text{ and } \mu < 0 \]

(3.19)

**Case g :** \( \theta > 0, \mu < 0 \), \( \rho = 2, \nu = -1, x(0) = 1, \dot{x}(0) = 0 \) from (3.19)

Now, equation (2.2) becomes

\[ \ddot{x} + 2x - x^3 = 0, x(0) = 1 \text{ and } \dot{x}(0) = 0 \]

which is an initial value problem, whether it is bounded or not and find its period.

**Solution :**

The solution is unbounded since \( \rho = 2, \nu = -1 \) and \( \omega_0 = \sqrt{\frac{\rho}{2}} \) which is positive.

Its period is \( \frac{4K(\sqrt{\frac{\rho}{2}})}{\sqrt{2}} \) from Jacobi function \( cd \).

**Case h :** \( \theta < 0 \) and \( \mu < 0 \)

Now, (3.16) becomes (see [9])

\[ x(t) = x_0 nd \left( \sqrt{-\theta(1 - \mu)} t, \frac{1}{\sqrt{1 - \mu}} \right), \theta < 0 \text{ and } \mu < 0 \]

(3.20)

**Case i :** \( \theta < 0 \), \( \mu < 0 \), \( \rho = -2, \nu = 13, x(0) = 10^{-2}, \dot{x}(0) = 0 \) from (3.20)

Now, equation (2.2) becomes

\[ \ddot{x} - 2x + 13x^3 = 0, x(0) = 10^{-2} \text{ and } \dot{x}(0) = 0 \]

which is a initial value problem, whether it is bounded or not and find its period.

**Solution :**

The solution is unbounded since \( \rho = -2, \nu = 13 \) and \( \omega_0 = 1.4139 \) which is positive.

Its period is \( 0.01 \times \frac{4K(0.99985)}{1.4139} \) from Jacobi function \( nd \).

**Case C : Solve Initial Value Problem for Linear \( \nu = 0 \)**

Equation (2.2) becomes \( \ddot{x}(t) + \rho x(t) = 0 \) which is a second order linear differential equation.

Now, solve the initial value problem \( \ddot{x}(t) + \rho x(t) = 0, x(0) = x_0, \dot{x}(0) = \dot{x}_0. \)

**Solution :** The auxiliary equation of \( \ddot{x}(t) + \rho x(t) = 0 \) is \( M^2 + \rho = 0 \)

\[ \Rightarrow M = 0 \pm i \sqrt{\rho} \]

\[ \therefore x(t) = e^{0 \times t} \left( c_1 \cos(\sqrt{\rho} t) + c_2 \sin(\sqrt{\rho} t) \right) \]

\[ = (c_1 \cos(\sqrt{\rho} t + c_2 \sin(\sqrt{\rho} t)) \]

\[ c_1, c_2 \text{ are constants.} \]

\[ \Rightarrow x(0) = (c_1 \cos(\sqrt{\rho} \times 0 + c_2 \sin(\sqrt{\rho} \times 0) \]

\[ \Rightarrow x_0 = c_1 \]

Now \( \dot{x}(t) = (-c_1 \sin(\sqrt{\rho} t + c_2 \cos(\sqrt{\rho} t) \]

\[ \Rightarrow \dot{x}(0) = 0 \]

\[ \Rightarrow \dot{x}_0 = c_2 = 0 \]

\[ \therefore x(t) = x_0 \cos(\sqrt{\rho} t) \]

is the required particular solution which is bounded and its period is to be determined by suitable value of \( \rho \).

**IV. GRAPHS OF VELOCITY AND ACCELERATION OF UNDAMPED AND UNDRIVEN DUFFING EQUATION USING**

**MATLAB** The concept of Duffing equation was named after Georg Duffing (1861 – 1944) which is a non-linear second order differential equation in the case of damped \( (\delta \neq 0) \) and driven \( (\gamma \neq 0) \) equation.

The duffing equation is of the form

\[ \ddot{x} + \delta \dot{x} + (\nu x^3 \pm \rho x) = \gamma \cos(\omega t) \]

(1.1)

where \( x(t) \) = displacement at time \( t \)

\[ \dot{x} = \frac{d}{dt}(x(t)) = \text{velocity} \]

and \( \ddot{x} = \frac{d^2}{dt^2}(x(t)) = \text{acceleration}. \)

The numbers \( \delta, \rho, \nu, \gamma, \omega \) are parameters.

**Note** that the motion of a damped \( (\delta \neq 0) \) and unforced / undriven \( (\gamma = 0) \) oscillator has more complex potential than Simple harmonic motion.

**C. Parameters**

The parameters of equation (1.1) are

- \( \delta \) depends on the amount of damping
- \( \rho \) depends on the linear case
- \( \nu \) depends on the non-linear case
- \( \gamma \) is the amplitude of the periodic driving force
- \( \omega \) is the angular frequency of the periodic driving force

**B. Equilibrium Points**

The force provided for the non-linear case is

\[ \nu x^3 + \rho x \]

(1.2)
Case – I
If $\rho > 0$ and $\nu > 0$ then (1.2) is called a hardening spring and the equilibrium point is at $x = 0$

Case – II
If $\rho > 0$ and $\nu < 0$ then (1.2) is called a softening spring and the equilibrium points are at $x = \pm \sqrt{\frac{\rho}{\nu}}$

Case - III
If $\rho < 0$ and $\nu > 0$ then (1.2) is also called a softening spring and the equilibrium points are at $x = \pm \sqrt{-\frac{\rho}{\nu}}$

C. Methods of Solution
Many approximate solutions for the duffing equation are
- By Fourier series method.
- By Frobenius method which yields a complex solution.
- By Euler’s method and Runge-Kutta methods in numerical analysis.
- By Homotopy analysis method which yields approximate solutions of the duffing equation.
- By Jacobi Elliptic functions to obtain the exact solutions of the undamped ($\delta = 0$) and undriven ($\gamma = 0$) duffing equation.

V. UNDAMPED ($\delta = 0$) AND UNFORCED ($\gamma = 0$) DUFFING EQUATION BY JACOBI ELLIPTIC FUNCTIONS

The duffing equation is

$$\ddot{x} + \delta \dot{x} + (\nu x^3 \pm \rho x) = \gamma \cos(\omega t)$$  \hspace{1cm} (2.1)

In this paper, my aim is to solve for boundedness and its time period of the duffing equation (undamped ($\delta = 0$) and unforced / undriven ($\gamma = 0$)) by Jacobi elliptic functions $cn$ and $nc$, $dn$ and $nd$, $cd$ and $dc$, $sn$. (see [2, 5-8, 11])

Also I expressed the identities, the properties and graphs using MATLAB program of Jacobi elliptic functions. (see [11])

By taking the plus sign in (2.1) becomes

$$\ddot{x} + \nu x^3 + \rho x = 0$$  \hspace{1cm} (2.2)

In this paper there are three special cases to solve for boundedness and time period. These are
(i) For cubic case ($\nu \neq 0$) (special case ($\nu = 1 = \rho$)) without initial conditions
(ii) For cubic case ($\nu \neq 0$) with initial conditions
(iii) For linear case ($\nu = 0$) with initial conditions

A. Properties of Jacobi Elliptic Functions
(i) Jacobi elliptic functions are $sn(t, m) = \sin \theta, cn(t, m) \cos \theta$, $dn(t, m) = \sqrt{1 - m^2 \sin^2 \theta}$

where $m \in (0, 1)$ is called elliptic modulus,

$$t = \int_0^\theta \frac{d\phi}{\sqrt{1 - m^2 \sin^2 \phi}}$$

is the time period

and $\theta = am(t, m)$ is called Jacobi amplitude.

(ii) Jacobi elliptic functions are derivable
(iii) Graphs using MATLAB (see [11])

Program:
>> m = 0.5;
>> t = -5 : 0.1 : 5;
>> [s, c, d] = ellipj(t, m);
>> plot(t, s, 's', t, c, '^', t, d, '.');
>> axis([-5 5 -1.1 1.1])
>> legend('sn', 'cn', 'dn')
>> xlabel('Time (t)')
>> ylabel('Elliptic Modulus')
>> title('Graph of Jacobi Elliptic Functions')

Graph:

(iv) $sn$ and $cn$ are periodic functions

period of $sn = 4K\left(\frac{1}{\nu}\right) = 4K(m)$, $m = \frac{1}{4}$

period of $cn = 4K\left(\frac{1}{\nu}\right) = 4K(m)$, $m = \frac{1}{4}$

where $K = K(m) = K\left(\frac{1}{\nu}\right) \approx 1.5962422$

B. Identities of Jacobi Elliptic Functions
- $sn^2(t, m) + cn^2(t, m) = 1$
- $dn^2(t, m) = 1 - m^2 sn^2(t, m)$
- \[ \lim_{m \to 0} sn(t, m) = sn(t, 0) \]
VI. SOLVE CUBIC FOR BOUNDEDNESS AND TIME PERIOD

Case A : For Cubic \( v \neq 0 \) (Special case \( \rho = 1 = v \))

From (1.1) the undamped \( (\delta = 0) \) and unforced \( (\gamma = 0) \) duffing equation is

\[
\ddot{x} + \rho \dot{x} + \nu x^3 = 0
\]

\[
\Rightarrow \ddot{x} + \rho \dot{x} + \nu x^3 = 0
\]

\[
\Rightarrow \frac{d}{dt} \left[ \frac{1}{2} (\dot{x})^2 + \frac{1}{2} \rho \dot{x}^2 + \frac{1}{4} \nu x^4 \right] = 0
\]

\[
\Rightarrow \frac{1}{2} (\dot{x})^2 + \frac{1}{2} \rho \dot{x}^2 + \frac{1}{4} \nu x^4 = \frac{H}{\rho}
\] (3.1)

is called invariant of motion and where \( H \) is called the Hamiltonian Operator. Where \( H \) is to be determined by putting the initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \). Here since \( \rho \) and \( \nu \) are positive then the solution of (3.1) is bounded.

If \( |x| \leq \sqrt{2H/\rho} \) and \( |\dot{x}| \leq \sqrt{2H} \) then the Hamiltonian \( H \) is positive.

Now Substituting \( \rho = 1 = \nu \) in (2.2) and (3.1),

we have \( \ddot{x} + x^3 + \dot{x} = 0 \) and \( \frac{1}{2} (\dot{x})^2 + \frac{1}{2} \dot{x}^2 + \frac{1}{4} x^4 = H \)

which transforms to Hamiltonian system of first order differential equations. (see [4], [10])

\[
\dot{x} = y
\]

\[
\dot{y} = -x - x^3
\]

and \( \frac{1}{2} (\dot{x})^2 + \frac{1}{2} \dot{x}^2 + \frac{1}{4} x^4 = H \) (3.4)

Now we have to find out the time period \( t \) by solving (3.4)

\[
(\frac{dx}{dt})^2 = 2H - x^2 - \frac{1}{2} x^4
\]

\[
\Rightarrow \frac{dx}{dt} = \sqrt{2H - x^2 - \frac{1}{2} x^4}
\]

\[
\Rightarrow dt = \int \frac{dx}{\sqrt{2H - x^2 - \frac{1}{2} x^4}} \text{ is the Time Period.}
\]

From (3.4)

\[
\frac{1}{2} y^2 + \frac{1}{2} x^2 + \frac{1}{4} x^4 = H(x, y)
\]

\[
\frac{\partial H}{\partial x} = x + x^3 = -\dot{y} = -\ddot{x}
\]

\[
\frac{\partial H}{\partial y} = y = \dot{x}
\]

The Hamiltonian system of equations (see [3]) are

\[
\dot{x} = \frac{\partial H}{\partial y}
\]

\[
\dot{y} = -\frac{\partial H}{\partial x}
\]

Case B : Solve Initial Value Problem (IVP) for Cubic with different Special Cases

Solve the initial value problem \( x(t) + \rho x(t) + \nu x^3(t) = 0 \), \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \) which is a second order non-linear differential equation by using Jacobi elliptic function.

Solution:

Setting \( x(t) = c_1 cn(\omega t + c_2, m) \) (3.5) satisfies the equation (2.2).

Now \( \omega \) and \( m \) are to be determined in terms of \( \rho \) and \( \nu \) and \( c_1, c_2 \) are to be determined from initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \). Now \( \dot{x}(t) + \omega^2(1 - 2m^2)x(t) + \frac{2m^2 \omega^2}{c_1^4} x^3(t) = 0 \),

\[
x(0) = x_0 \text{ and } \dot{x}(0) = \dot{x}_0
\] (3.6)

where \( c_1, c_2 \) are constants.

Now, comparing (2.2) and (3.6), we get

\[
\rho = \omega^2(1 - 2m^2) \text{ and } \nu = \frac{2m^2 \omega^2}{c_1^4}
\]

\[
\Rightarrow \rho = \omega^2 - 2m^2 \omega^2 \text{ and } vc_1^2 = 2m^2 \omega^2
\]

\[
\Rightarrow \omega = \sqrt{\rho + vc_1^2}
\] (3.7)

and \( vc_1^2 = 2m^2 \omega^2 \)

\[
\Rightarrow vc_1^2 = 2m^2 (\rho + vc_1^2) \text{ (\( \omega^2 = \rho + vc_1^2 \))}
\]

\[
\Rightarrow \frac{m^2}{2(\rho + vc_1^2)} = \frac{vc_1^2}{2(\rho + vc_1^2)}
\]

\[
\Rightarrow \frac{m = \frac{vc_1^2}{\sqrt{2(\rho + vc_1^2)}}}{\sqrt{2(\rho + vc_1^2)}}
\] (3.8)

Now, substituting the values of (3.7) and (3.8) in (3.5), we get

\[
x(t) = c_1 cn\left(\sqrt{\rho + c_1^4}t + c_2, \frac{c_1^4 \nu}{2(\rho + c_1^4) v}\right)
\]

\[
\Rightarrow x(0) = c_1 cn\left(\sqrt{\rho + c_1^4} \times 0 + c_2, \frac{c_1^4 \nu}{2(\rho + c_1^4) v}\right)
\]

\[
= c_1 cn(c_2, m)
\]

\[
= x_0
\]
and \( \dot{x}(0) = c_1 \left[ -\frac{\sin \left( \sqrt{\rho + c_1^2 v} \right) \times 0 + c_2, m) d(c_2, m) \right] \frac{1}{\sqrt{\rho + c_1^2 v}} \) 

\( = -\sqrt{\rho + c_1^2 v} c_1 \sin(c_2, m) d(c_2, m) \)

\( = x_0 \)

Now I have to discuss about the solution by different cases to the given problem in the form of \( x(t) = x_0 \phi(\omega_0 t, m_0) \) in terms of Jacobi elliptic functions nc, dc and cd, nd having positive frequency \( \omega_0 \) and modulus \( m_0 \) on the interval \([0, 1]\).

**Case i :** \( \dot{x}(0) = 0 \)

Then \( -\sqrt{\rho + c_1^2 v} c_1 \sin(c_2, m) d(c_2, m) = 0 \)

\( \Rightarrow c_1 = 0 = x_0 \quad (3.10) \)

\( \therefore c_2 = 0 \quad (3.11) \)

Now the solution to the initial value problem

\( \ddot{x}(t) + \rho \dot{x}(t) + \nu x^3(t) = 0, \quad x(0) = x_0 \) and \( \dot{x}(0) = 0 \)

(3.12)

is

\( x(t) = x_0 cn \left( \sqrt{\rho + \nu x_0^2} t, \sqrt{\frac{\nu x_0^2}{\sqrt{\rho + \nu x_0^2}}} \right), \quad \rho + \nu x_0^2 \neq 0 \)

(3.13)

Here the required solution is bounded.

Substituting \( \rho = 1 = \nu \) in (3.12), we get the time period.

**Case ii :** \( x(0) = 1, \dot{x}(0) = 0, \rho = \nu = 2 \)

Now, equation (2.2) becomes

\( \ddot{x} + 2x + 2x^3 = 0, \quad x(0) = 1 \) and \( \dot{x}(0) = 0 \)

which is a initial value problem, whether it is bounded or not and find its period.

**Solution :**

Now, \( \theta = (-2) + (-2)^2 = \mu = \frac{(-2)^2}{2(-2+(-2)^2)} \)

\( = -4 \quad \text{and} \quad \mu = \frac{1}{4} \)

\( \therefore x(t) = nc \left( \sqrt{-4} t, \sqrt{1 - \frac{1}{4}} \right) \)

\( = nc \left( 2t, \frac{\sqrt{3}}{2} \right) \)

Again, \( x(t) = 1 \times cn \left( \sqrt{(-2) + (-2)^2} t, \sqrt{\frac{(-2)^2}{2((-2)+(-2)^2)}} \right) \)

\( = cn \left( 2\sqrt{-1} t, \frac{1}{2} \right) \)

\( \therefore x(t) = nc \left( 2t, \frac{\sqrt{3}}{2} \right) = cn \left( 2\sqrt{-1} t, \frac{1}{2} \right) \)

The solution is unbounded since \( \nu = \rho = 2 \) and \( \omega_0 = 2\sqrt{-1} \) which is negative.

Its period is \( \frac{4k(\sqrt{3})}{2} \) from Jacobi function \( nc \).

**Case c :** \( \mu = 1, \rho = 1, \nu = -2, x(0) = 1, \dot{x}(0) = 0 \)

from (3.17)

Now, equation (2.2) becomes

\( \ddot{x} + x - 2x^3 = 0, \quad x(0) = 1 \) and \( \dot{x}(0) = 0 \)

which is a initial value problem, whether it is bounded or not and find its period.

**Solution :**

The solution is unbounded since \( \nu = -2, \rho = 1 \) and \( \omega_0 = \sqrt{-1} \) which is negative.

Its period is \( 4K(0) \) from Jacobi function \( nc \).

**Case d :** \( \theta < 0 \) and \( \mu > 1 \)

Now, (3.16) becomes (see [2])

\( x(t) = x_0 dc \left( \sqrt{-\theta \mu} t, \sqrt{1 - \frac{1}{\mu}} \right), \theta < 0 \) and \( \mu > 1 \)

(3.18)

**Case e :** \( \theta < 0, \mu > 1, \rho = 3, \nu = -1, x(0) = 2, \dot{x}(0) = 0 \)

from (3.18)

Now, equation (2.2) becomes

\( \ddot{x} + 3x - x^3 = 0, \quad x(0) = 2 \) and \( \dot{x}(0) = 0 \)

which is an initial value problem, whether it is bounded or not and find its period.

**Solution :**

The solution is unbounded since \( \rho = 3, \nu = -1 \) and \( \omega_0 = \sqrt{3} \) which is positive.

Its period is \( 2 \times \frac{4k(\sqrt{3})}{2} \) from Jacobi function \( dc \).
Case \( f \) : \( \vartheta > 0 \) and \( \mu < 0 \)
Now, \( (3.16) \) becomes (see [9]) \[
x(t) = x_0 \text{cd} \left( \sqrt{(1 - \mu)t}, \frac{\sqrt{\rho}}{\sqrt{1 - \mu}} \right), \quad \vartheta > 0 \quad \text{and} \quad \mu < 0
\]
\[
(3.19)
\]

Case \( g \) : \( \vartheta > 0 \), \( \mu < 0 \), \( \rho = 2 \), \( \nu = -1 \), \( x(0) = 1 \), \( \dot{x}(0) = 0 \)
Now, equation \( (2.2) \) becomes \[
\ddot{x} + 2x - x^3 = 0, \quad x(0) = 1 \quad \text{and} \quad \dot{x}(0) = 0
\]
which is an initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since \( \vartheta = 2 \), \( \nu = -1 \) and \( \omega_0 = \sqrt{2} \) which is positive.

Its period is \( \frac{4\pi}{\sqrt{2}} \) from Jacobi function \( cd \).

Case \( h \) : \( \vartheta < 0 \) and \( \mu < 0 \)
Now, \( (3.16) \) becomes (see [9]) \[
x(t) = x_0 \text{nd} \left( \sqrt{-\vartheta(1 - \mu)t}, \frac{1}{\sqrt{1 - \mu}} \right), \quad \vartheta < 0 \quad \text{and} \quad \mu < 0
\]
\[
(3.20)
\]

Case \( i \) : \( \vartheta < 0 \), \( \mu < 0 \), \( \rho = -2 \), \( \nu = 13 \), \( x(0) = 10^{-2} \), \( \dot{x}(0) = 0 \)
Now, equation \( (2.2) \) becomes \[
\ddot{x} - 2x + 13x^3 = 0, \quad x(0) = 10^{-2} \quad \text{and} \quad \dot{x}(0) = 0
\]
which is a initial value problem, whether it is bounded or not and find its period.

Solution :

The solution is unbounded since \( \rho = -2 \), \( \nu = 13 \) and \( \omega_0 = 1.4139 \) which is positive.

Its period is \( 0.01 \times \frac{4\pi(0.99985)}{1.4139} \) from Jacobi function \( nd \).

Case \( C \) : Solve Initial Value Problem for Linear \( \nu = 0 \)
Equation \( (2.2) \) becomes \( \ddot{x}(t) + px(t) = 0 \) which is a second order linear differential equation.
Now, solve the initial value problem \( \ddot{x}(t) + px(t) = 0 \), \( x(0) = x_0 \), \( \dot{x}(0) = \dot{x}_0 \).

Solution : The auxiliary equation of \( \ddot{x}(t) + px(t) = 0 \) is \( M^2 + \rho = 0 \)
\[
\Rightarrow M = 0 \pm i \sqrt{\rho}
\]
\[
\therefore x(t) = e^{\omega t}(c_1 \cos \sqrt{\rho} t + c_2 \sin \sqrt{\rho} t)
\]
\[
= (c_1 \cos \sqrt{\rho} t + c_2 \sin \sqrt{\rho} t),
\]
c1, c2 are constants.
\[
\Rightarrow x(0) = (c_1 \cos \sqrt{\rho} \times 0 + c_2 \sin \sqrt{\rho} \times 0)
\]
\[
\Rightarrow x_0 = c_1
\]
Now \( \dot{x}(t) = -c_1 \sin \sqrt{\rho} t + c_2 \cos \sqrt{\rho} t \)
\[
\Rightarrow \dot{x}(0) = 0
\]
\[
\Rightarrow \dot{x}_0 = c_2 = 0
\]
\[
\therefore x(t) = x_0 \cos \sqrt{\rho} t
\]
\[
(3.21)
\]
is the required particular solution which is bounded and its period is to be determined by suitable value of \( \rho \).

IV. GRAPHS OF VELOCITY AND ACCELERATION OF UNDAMPED AND UNDRIVEN DUFFING EQUATION USING MATLAB
Consider the duffing equation
\[
\ddot{x} + \delta \dot{x} + (\nu x^3 \pm \rho x) = \gamma \cos(\omega t)
\]
(4.1)
where \( x = x(t), t = time \ period \)
Here \( x(t) \) = displacement, \( \dot{x} \) = velocity, \( \ddot{x} \) = acceleration
The graphs of velocity and acceleration for duffing equation with suitable parameters \( \rho = -2, \nu = 13, \gamma = 0, \delta = 0, \omega = 1.4139 \) using MATLAB (see [1]) program are shown below where X-axis indicates Time (t) in second and Y-axis indicates Velocity and Acceleration Graph with respect to Time (t).

Program : M-file

function xdiff = duffing_2(t,x)
rho = -2;
nu = 13;
gamma = 0;
delta = 0;
omega = 1.4139;
dxdt = [x(2); -delta*x(2)-rho*(x(1))-nu*(x(1)^3)+gamma*cos(omega*t)];
xdiff = dxdt;
return

Command Window
>> [t,x] = ode45(@duffing_2, [0 2], [0 1]);
>> plot(t, x, '+')
>> title('DUFFING EQUATION : \rho = -2, \nu = 13, \gamma = 0, \delta = 0, \omega = 1.4139')
>> xlabel('Time (t) in second')
>> ylabel('Velocity and Acceleration')
>> legend('Velocity', 'Acceleration')
Consider the duffing equation
\[ \ddot{x} + \delta \dot{x} + (\nu x^3 \pm \rho x) = \gamma \cos(\omega t) \quad (4.1) \]
where \( x = x(t), t = \text{time period} \)

Here \( x(t) = \text{displacement}, \ \dot{x} = \text{velocity}, \ \ddot{x} = \text{acceleration} \)

The graphs of velocity and acceleration for duffing equation with suitable parameters \( \rho = -2, \nu = 13, \gamma = 0, \delta = 0, \omega = 1.4139 \) using MATLAB (see [1]) program are shown below where X-axis indicates Time (t) in second and Y-axis indicates Velocity and Acceleration Graph with respect to Time (t).

**Program:**

**M-file**

function xdiff = duffing_2(t,x)
rho = -2;
nu = 13;
gamma = 0;
delta = 0;
omega = 1.4139;
xdot = [x(2); -delta*x(2)-rho*(x(1))-nu*(x(1)^3)+gamma*cos(omega*t)];
xdiff = xdot;
return

**VII. REFERENCES**

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